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Rotation Numbers of Projectivities

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Let D be a nonempty open convex region in $\mathbb{R}^2$, bounded by a closed curve C. For any $a \in D$ we define a map $f_a: C \to C$ by requiring that for each $x \in C$, the point $f_a(x)$ be the point of C other than x which lies on the line through x and a. If $A := (a_1, a_2, \ldots, a_n)$, where $a_1, a_2, \ldots, a_n \in D$, then we set $f_A := f_{a_n} \circ f_{a_{n-1}} \circ \cdots \circ f_{a_1}$. Since it is an orientation preserving homeomorphism of C onto itself and C is homeomorphic to a circle, we can speak about the rotation number of f_A, which we denote by $\rho(A)$. We investigate the dynamical properties of f_A depending on C and $\rho(A)$. In particular, we show that if $n \geq 3$ and $k \geq 1$ are integers and $\rho(A) = (2k+1)/2$ for some $A \in D^n$, then $f_A^2 = \mathrm{id}$ for all $A \in \rho^{-1}((2k+1)/2)$ if and only if C is an ellipse.

1. INTRODUCTION

Given two lines L and M in the plane, it is customary in projective geometry to define a map from L to M by fixing a point a in the plane, not on L or M, and letting $x \in L$ be mapped to $y \in M$ when x, y, and a are collinear. Such a map is called a *perspectivity*, and a composition of perspectivities is called a *projectivity*. In this article we define and study analogous maps from a closed curve C onto itself. Specifically, let D be a nonempty open convex region in $\mathbb{R}^2$ bounded by a closed curve C. For any $a \in D$ we can define a homeomorphism $f_a: C \to C$ by requiring that for each $x \in C$, the point $f_a(x)$ be the point of C other than x which lies on the line through x and a. Borrowing the above terminology, let us call f_a the *perspectivity* of C onto itself with center a. Similarly, if $A \in D^n$ is defined by $A := (a_1, a_2, \ldots, a_n)$, let us refer to the homeomorphism $f_A: C \to C$ defined by $f_A := f_{a_n} \circ f_{a_{n-1}} \circ \cdots \circ f_{a_1}$ as the *projectivity* of C onto

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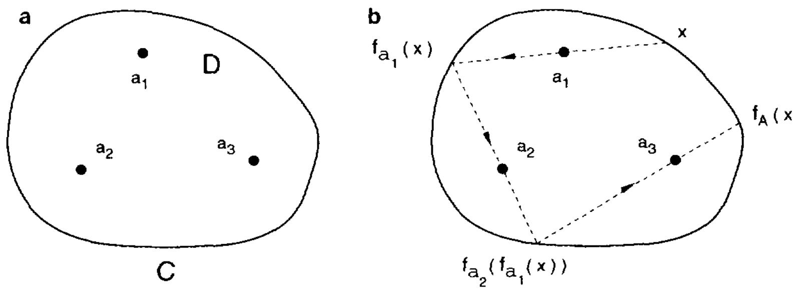

FIG. 1. (a), (b). Constructing a projectivity.

itself via the sequence $a_1, a_2, ..., a_n$. As an example of such a system, let C be an ellipse with focal points a_1 and a_2, and let $A := (a_1, a_2)$. Then the sequence $\{f_A^k(x)\}_{k=1}^{\infty}$ for an arbitrary point $x \in C$ is the sequence of reflection points for a corresponding light ray which is reflected inside the ellipse from focus to focus, so this case is actually a billiards problem. It is not hard to show that any such ray must approach a trajectory which is coincident with the major axis, and it was recently shown in [3] that this leads to a conveniently describable wave-intensification property for perfectly reflecting ellipsoids. Another application of such projectivities is the determination of the existence or nonexistence of certain polygons inscribed in D. For example, given the convex region D with boundary C in Fig. 1a, with $a_1, a_2, a_3 \in D$, one can ask, "Does there exist a triangle inscribed in C, each of whose sides contains one of the a_i?"

If we let $A := (a_1, a_2, a_3)$, this question is equivalent to asking, "Does the projectivity f_A have a fixed point $x_0 \in C$?" One can guess the answer in the above case by choosing an arbitrary point $x \in C$ and using a pencil and straightedge to iteratively apply f_A to x, as begun in Fig. 1b. The reader is urged to try this in Fig. 1a (or a photocopy!). Eventually, the line segments being sketched will "converge" to an inscribed triangle with the desired property, such as the previously mentioned light ray in an ellipse converges to a line segment. Further experimentation reveals that if some of the a_i are moved slightly, then the convergence to a triangle still occurs. This type of stability is sometimes referred to as "phase locking." On the other hand, it is possible to place a_1, a_2, and a_3 in D (grouped near the center, for example) in such a way as to preclude the existence of such a triangle. In this case one often finds that the points $f_A^k(x)$ seem to be wandering all over C, suggesting that the orbit $\{f_A^k(x)\}_{k=1}^{\infty}$ is dense in C.

More generally, we would like to describe the dynamical behavior of the projectivity $f_A: C \to C$ for various choices of $a_1, a_2, ..., a_n$. It is clear that each f_{a_i}, and hence f_A, is an orientation preserving homeomorphism of C, and that C is homeomorphic to the circle S^1 via an orientation preserving homeomorphism. We can then consider f_A, for each $A \in D^n$, as a circle homeomorphism, and in particular we can speak of the rotation number of f_A, which we will denote by $\rho(A)$.[1] Now each perspectivity f_{a_i} depends on a_i in a continuous way and, hence, the projectivity f_A depends on A in a continuous way. It is also known that the dependence of the rotation number on a homeomorphism is continuous. Thus we get a continuous function $\rho : D^n \to \mathbb{R}$. On the other hand, we should *not* think of $\rho(A)$ as being exclusively determined by f_A, without reference to the component perspectivities f_{a_i}. For example, suppose A and A' are defined by $A := (a, a)$ and $A' := (a, a, a, a)$ for some point $a \in D$. We think of f_A as taking a point $x \in C$ once anticlockwise around C and we think of $f_{A'}$ as taking the same point *twice* anticlockwise around C, so that $\rho(A) = 1$ and $\rho(A') = 2$. Nevertheless, technically speaking, f_A and $f_{A'}$ are the same function. We will give a precise definition of $\rho(A)$ in the next section.

It will be advantageous to study the projectivities f_A by studying the function ρ. For example, if $\rho(A) = p/q$ for coprime positive integers p and q, then there exists a periodic point x of f_A with period q and a corresponding qn-gon P inscribed in C, such that $\{a_1, a_2, ..., a_n\} \subset P$, and each side of P contains one of the a_i. (We will use the expression "k-gon" to mean a union $x_0x_1 \cup x_1x_2 \cup ... \cup x_{k-2}x_{k-1} \cup x_{k-1}x_k$ of line segments $x_{i-1}x_i$ in the plane with endpoints x_{i-1}, x_i, such that $x_k = x_0$.) If $q > 1$, then P is self-crossing and passes through each a_i at least q times. If $\rho(A)$ is irrational, then f_A is conjugate to an irrational rotation, provided that C is piecewise C^2 with a finite number of pieces (from Lemma 10 it follows that f_A is also piecewise C^2, and then we can use the Denjoy theorem; it is known that its proof also works under those assumptions). In this case, we see that the orbit of any point $x \in C$ is indeed dense in C as hinted at above.

Note that if the entire picture in Fig. 1b is subjected to a homeomorphism P which preserves straight lines and the orientation on C (a projective transformation), and if the image $D' := P(D)$ is bounded, with boundary C', then the images a_i' of the a_i can be used, in the obvious way, to define a projectivity $f_{A'} : C' \to C'$. Moreover, $f_{A'}$ will have the same rotation number as f_A. In particular, this is true when P is an affine transformation with a positive determinant. The resulting convenience is that properties of projectivities which can be proved for a simply shaped region (e.g., an equilateral triangle or a circle) can be immediately generalized (e.g., to an arbitrary triangle or an ellipse).

The paper is organized as follows. In Section 2 we determine all possible

[1] Rotation numbers were introduced by Poincaré (see [5]). One can find some information on them in virtually every introductory book on dynamical systems (see, e.g., [4, 6, 1]).

rotation numbers for f_A, where A consists of n points. It turns out to be the interval $[1, n - 1]$, unless C is a polygon. In this case the result is more complicated.

In Section 3 we consider projectivities on ellipses. As we noted already, we can subject the whole picture to an orientation preserving affine transformation, and thus we can assume that C is the unit circle. In this case the class of all projectivities is equal to the class of all orientation preserving Möbius transformations. We prove that unless $\rho(A)$ is an integer, f_A is conjugate to a rotation, and we present a method of computing $\rho(A)$.

In Section 4 we are concerned with the question of whether a certain property (namely, that projectivities with non-integer rotation numbers are conjugate to rotations) characterizes ellipses. We prove that for every $n \geq 3$ and $k \geq 1$ if every projectivity on C via n points with rotation number $(2k + 1)/2$ is conjugate to a rotation then C is an ellipse. We conjecture that the same is true with $(2k + 1)/2$ replaced by any non-integer rational number. We prove also that for any integer k and any C and n, the set of projectivities via n points with rotation number k has non-empty interior (unless there is no projectivity with this rotation number).

2. The Range of ρ

Circle homeomorphisms that are conjugate via an orientation preserving homeomorphism have the same rotation number. Therefore ρ does not depend on the choice of homeomorphism between C and S^1, so long as it preserves orientation. However, $\rho(A)$ is defined only up to a shift by an integer. Note that since D^n is connected, a different version of $\rho: D^n \to \mathbb{R}$ will be continuous if and only if it is obtained by adding the same integer to all $\rho(A)$.

To specify which version of ρ we use, we make the following construction. Fix some orientation preserving homeomorphism $h: S^1 \to C$. If $\tilde{e}: \mathbb{R} \to S^1$ is the natural projection $\tilde{e}(t) = \exp(2\pi i t)$, then we will use $e := h \circ \tilde{e}$ to construct liftings of homeomorphisms of C and to measure distances between the points of C. The distances will be measured along C in the anticlockwise direction. Thus if $x, y \in C$, then the distance from x to y is $d(x, y) := s - t$, where $t \in e^{-1}(x)$ and $s := \min\{u \in \mathbb{R} | u \in e^{-1}(y) \text{ and } u \geq t\}$. Note that this distance is not symmetric; in general, $d(x, y) \neq d(y, x)$. However, such a situation is not unusual. After all, there are five days from Friday till Wednesday, but only two days from Wednesday till Friday. Now we note that for each $a \in D$ and $x \in C$ we have $f_a(x) \neq x$. Thus there exists a unique lifting F_a of f_a such that $0 < F_a(t) - t < 1$ for all $t \in \mathbb{R}$. Such a choice is also continuous with respect to a. Note that with our choice, $F_a(t) - t = d(x, f_a(x))$, where $x = e(t)$. At last we take $F_A :=$

$F_{a_n} \circ F_{a_{n-1}} \circ \cdots \circ F_{a_1}$ for $A = (a_1, a_2, ..., a_n)$. Then $\rho(A) := \lim_{n\to\infty} (F_A^n(t) - t)/n$ for any $t \in \mathbb{R}$.

We want to find the image of D^n under ρ. That is, we want to know how much the rotation number can change when we vary the configuration of the points $a_1, a_a, ..., a_n$. We already know that $0 < F_a(t) - t < 1$ for all $a \in D$ and $t \in \mathbb{R}$, so $0 < F_A(t) - t < n$ for all $A \in D^n$ and $t \in \mathbb{R}$. This gives an estimate $0 < \rho(A) < n$. We can, in fact, determine $\rho(D^n)$ precisely. To do so, it turns out that we must first specify whether or not C is a polygon. The reason for this is connected to whether or not we can inscribe a convex n-gon into C (by "inscribe" we mean that only the vertices of the n-gon meet C; "convex" has the obvious meaning). In this regard, the following lemma will be useful. The proof is easy and is omitted.

LEMMA 1. *Let D be a nonempty open convex region in $\mathbb{R}^2$ bounded by a closed curve C, and let $n \geq 2$ be a positive integer. If C is not a polygon, or if C is a polygon with at least n sides, then we can inscribe a convex n-gon into C.*

We will use another simple fact.

LEMMA 2. *Let $F: \mathbb{R} \to \mathbb{R}$ be a lifting of an orientation preserving circle homeomorphism and let N be an integer. Assume that there exists $t \in \mathbb{R}$ such that $F(t) - t > N$. Then the rotation number of F is greater than or equal to N.*

Proof. Suppose that the rotation number of F is smaller than N. Then there is a point $s \in \mathbb{R}$ such that $F(s) - s < N$. Therefore between t and s there is a point u such that $F(u) - u = N$, and thus the rotation number of F is N, a contradiction. ∎

We are now ready to determine $\rho(D^n)$, beginning with the case when C is not a polygon.

THEOREM 3. *Let $n \geq 2$ and suppose that C is not a polygon, or that C is a polygon with at least n sides. Then $\rho(D^n) = [1, n - 1]$.*

Proof. There is a point $x \in C$ such that x, a_1, and a_2 lie on a straight line. Then $f_{a_1}(x) \neq x$ and $f_{a_2} \circ f_{a_1}(x) = x$. Thus, $d(x, f_{a_1}(x)) + d(f_{a_1}(x), f_{a_2} \circ f_{a_1}(x)) = 1$, so $F_{a_2} \circ F_{a_1}(t) - t = 1$ for each $t \in e^{-1}(x)$. Consequently, $F_A(t) - t \geq 1$ for such t and by Lemma 2 we get $\rho(A) \geq 1$. We can apply the same argument with the distances along C measured in the clockwise direction (alternatively, one can think of replacing C by its mirror image). It is easy to see that the rotation number obtained in such a way is n minus the old rotation number. Hence, $\rho(A) \leq n - 1$. Therefore we get $\rho(D^n) \subseteq [1, n - 1]$.

By Lemma 1, we can inscribe into C a convex polygon with n sides. We denote its consecutive vertices by $x_0, x_1, ..., x_{n-1}$. Let us also write $x_n :=$ x_0. Then choose points $a_1, a_2, ..., a_n$ on the sides $x_0x_1, x_1x_2, ..., x_{n-1}x_n$, respectively. We have $x_i = f_{a_i}(x_{i-1})$ for $i = 1, 2, ..., n$. If the ordering of the vertices is $x_0, x_1, ..., x_{n-1}$ when we move in the anticlockwise direction, then $\sum_{i=1}^{n} d(x_{i-1}, x_i) = 1$ and therefore $F_A(t) - t = 1$ for $t \in e^{-1}(x)$. Thus $\rho(A) = 1$. If we get this ordering when we move in the clockwise direction, then $\sum_{i=1}^{n} d(x_{i-1}, x_i) = n - 1$ and therefore $F_A(t) - t = n - 1$ for $t \in e^{-1}(x)$. Thus $\rho(A) = n - 1$. Since ρ is continuous and D^n is connected, this proves that $\rho(D^n) \supseteq [1, n - 1]$. Consequently, $\rho(D^n) = [1, n - 1]$. ▮

In the next theorem, we use the notation $E(r)$ to denote the integer part of a real number r.

THEOREM 4. *If C is a k-gon and $n \geq 2$, then $\rho(D^n) = [m, n - m]$, where $m = 1 + E((n - 1)/k)$ if k does not divide $n - 1$, and $m = \frac{1}{2} + (n - 1)/k$ if k divides $n - 1$.*

Proof. Let us fix our notation. If the sequence of n points is $A = (a_1, ..., a_n)$, then for a starting point x_0 we set $x_1 := f_{a_1}(x_0)$, $x_2 := f_{a_2}(x_1)$, ..., $x_n := f_{a_n}(x_{n-1})$, $x_{n+1} := f_{a_1}(x_n)$, ... The corresponding points in the lifting will be denoted X_i. The consecutive sides of the k-gon C will be denoted by $s_0, s_1, ..., s_{k-1}$, and the corresponding intervals in the lifting will be S_0, $S_1, ..., S_{k-1}, S_k, ...$.

If $a_1 = a_2 = \cdots = a_n$, then it is easy to see that $\rho(A) = n/2$. If k does not divide $n - 1$, then, since $k \geq 3$, we have $\rho(A) = 1 + (n - 2)/2 \geq 1 + (n - 2)/k \geq 1 + E((n - 1)/k)$. In any case, we also have $\rho(A) = \frac{1}{2} + (n - 1)/2 > \frac{1}{2} + (n - 1)/k$. Thus to show that ρ cannot be less than the minimum value m given above, we may assume that the a_i are not all equal. Moreover, if $A' \in D^n$ is obtained from A by a cyclic permutation of the indices, then $\rho(A') = \rho(A)$, so we may further assume for convenience that $a_1 \neq a_2$.

The ray from a_2 through a_1 intersects C at a point z. Fix some lifting Z of z. We can choose s_0 in such a way that $z \in s_0$ and S_0 such that $Z \in S_0$. In the case when z is a vertex, we require that Z be the left-hand endpoint of S_0. Choose X_0 to be the right-hand endpoint of S_0. Then $X_2 \in S_{i_2}$ with $i_2 \geq k + 1$; i.e., $f_{a_2} \circ f_{a_1}$ advances x_0 anticlockwise by at least $k + 1$ sides (see Fig. 2). Therefore, since each perspectivity f_{a_i} must advance a point of C by at least one side, for $j \geq 2$ we have $X_j \in S_{i_j}$ with $i_j \geq k + j - 1$. In particular, $X_n \in S_{i_n}$ with $i_n \geq k + n - 1$, and $X_{2n} \in S_{i_{2n}}$ with $i_{2n} \geq k + 2n - 1$.

If k does not divide $n - 1$, then $i_n \geq k + n - 1 > k(1 + E((n - 1)/k))$, so $X_n \geq X_0 + 1 + E((n - 1)/k)$. Thus, by Lemma 2, $\rho(A) \geq 1 + E((n - 1)/k)$. If k divides $n - 1$, then $i_{2n} \geq k + 2n - 1 > k(1 + 2(n - 1)/k)$, so $X_{2n} \geq X_0 + 1 + 2(n - 1)/k$. Thus, again by Lemma 2, $\rho(A) \geq \frac{1}{2} + (n - 1)/k$.

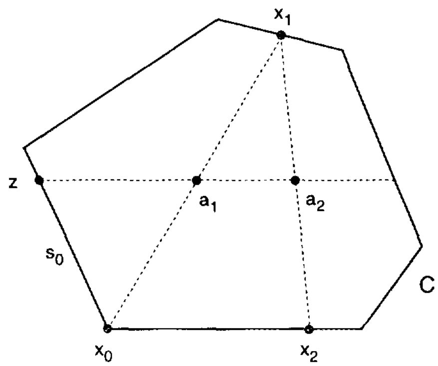

FIG. 2. Projectivity on a polygon.

Now we need examples of configurations with those minimal numbers. If $n \leq k$, then by Theorem 3 there exists an A such that $\rho(A) = 1 = 1 + E((n-1)/k)$, so let us assume that $n > k$. We choose x_i in the interior of s_i for $i = 0, 1, \ldots, k-1$ and then we define by induction $x_i := x_{i-k}$ for $k \leq i \leq n-1$. If k does not divide $n-1$, then we choose a_i in the interior of the chord $x_{i-1}x_i$ for $i = 1, 2, \ldots, n-1$ and choose a_n in the interior of the chord $x_{n-1}x_0$ so that $x_n = x_0$ (cf. Fig. 3a). This is consistent with the definition of the x_i's and we get $\rho(A) = 1 + E((n-1)/k)$.

If k divides $n-1$, then we go on and choose x_{n+i} in the interior of s_{i+1} for $i = 0, 1, \ldots, k-1$ (we consider $s_k \equiv s_0$), and then define by induction $x_{n+i} := x_{n+i-k}$ for $k \leq i \leq n-1$. We do it in such a way that the ordering

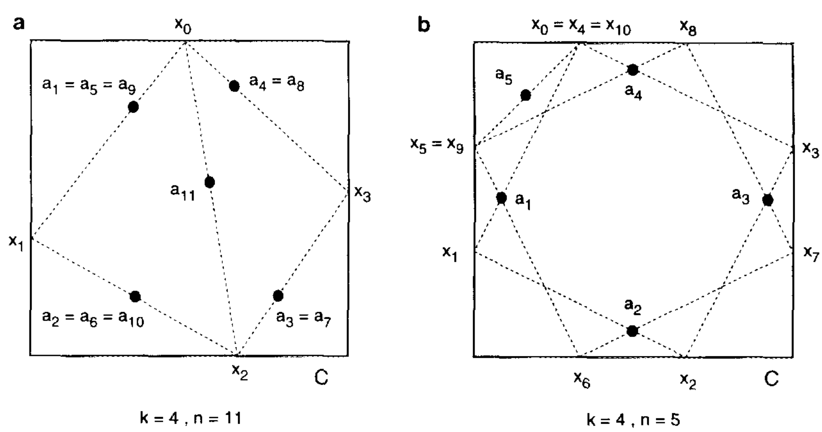

FIG. 3. (a), (b). Examples with minimal rotation numbers.

of the points along C is $x_0, x_n, x_1, x_{n+1}, \ldots, x_{k-1}, x_{n+k-1}$. Now we choose a_i as the point of intersection of the chords $x_{i-1}x_i$ and $x_{n+i-1}x_{n+i}$ for $i = 1, 2, \ldots, n-1$ and in the interior of the chord x_0x_n for $i = n$ (note that $x_0 = x_{n-1} = x_{2n}$; cf. Fig. 3b). This is consistent with the definition of the x_i's and we get $\rho(A) = \frac{1}{2} + (n-1)/k$.

Having established the minimal rotation numbers m, we can verify that the maximal rotation numbers are $n - m$ by measuring distances along C in the clockwise direction as was done in the proof of Theorem 3. Finally, since D^n is connected, we have $\rho(D^n) = [m, n-m]$. ∎

3. Circles and Ellipses

It turns out that $\rho(A)$ can be computed when C is an ellipse. Given an equation for C, the first step is to affinely project C onto the unit circle S^1, and to map the points $a_1, a_2, \ldots, a_n$ into its interior by the same affine map. As we mentioned in the introduction, the rotation number for the corresponding projectivity on the circle will be the same as for the ellipse. It is easiest to work with complex numbers, so from now on we assume that $C = S^1$ is the unit circle in $\mathbb{C}$, and for each j we have $a_j \in \mathbb{C}$ with $|a_j| < 1$. It is straightforward to verify that any perspectivity $f_a : S^1 \to S^1$ with center a can be written as a Möbius transformation $f_a(z) = (z - a)/(\bar{a}z - 1)$, where $\bar{a}$ denotes the conjugate of a. (It is worth recalling a familiar result of complex variable theory which says that every orientation preserving Möbius transformation T of S^1 onto itself can be written $T(z) = e^{i\theta}(z - a)/(\bar{a}z - 1)$, where $|a| < 1$ and $\theta \in \mathbb{R}$. Thus in our terminology, every orientation preserving Möbius transformation of S^1 onto itself is a perspectivity followed by a rotation.) Obviously f_a can also be written in the form $f_a(z) = (iz - ia)/(i\bar{a}z - i)$, and our reason for doing this is that we wish to consider Möbius transformations $h_{u,v}$ of the form

$$h_{u,v}(z) = \frac{uz + v}{\bar{v}z + \bar{u}},$$

where $|v| < |u|$. Under these restrictions, it is known that $h_{u,v}|_{S^1}$ is an orientation preserving homeomorphism of S^1 onto itself and, as just implied, whenever a Möbius transformation takes S^1 onto itself and preserves orientation, it can be written in the above form. We will write

$$h_{u,v} \sim \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix},$$

in order to make use of the well-known fact that

$$h_{u,v} \circ h_{r,s} \sim \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix}.$$

Note that such matrix representations are not unique, for a nonzero real multiple of such a matrix has the same form and represents the same transformation. If two matrices M and N represent the same transformation, we will write $M \sim N$.

The composition of Möbius transformations is a Möbius transformation, so f_A can be written as a transformation of the type $h_{u,v}$. Thus our problem boils down to that of determining the rotation numbers of the maps $h_{u,v}|_{S^1}$. Let us begin by finding the fixed points of $h_{u,v}$. To do so, we solve the quadratic equation $uz + v = z(\bar{v}z + \bar{u})$ for z to get

$$z = \frac{i\,\mathrm{Im}(u) \pm \sqrt{|v|^2 - (\mathrm{Im}(u))^2}}{\bar{v}}.$$

It is straightforward to check that $|z| = 1$ if and only if $|v| \geq |\mathrm{Im}(u)|$, and this is precisely when $h_{u,v}|_{S^1}$ has fixed points. Then $\rho(A)$ is an integer. From now on we will assume that $|v| < |\mathrm{Im}(u)|$. It is well known that if $h_{u,v}|_{S^1}$ has no fixed point then it is conjugate to rotation. Since this is an important fact, we state it as a theorem (in fact the computations following the theorem give an alternative proof).

THEOREM 5. *If C is an ellipse and $\rho(A)$ is not an integer, then f_A is conjugate to a rotation. In particular, if $\rho(A) = p/q$ with p and q coprime and $q \neq 1$ then f_A^q* = id.

In order to find the rotation number of $f_A = h_{u,v}$, let us assume that $v \neq 0$ (the case $v = 0$ is easy; we will do it later) and consider the function $g := h_{\bar{w},iwx} \circ h_{u,v} \circ h_{w,-iwx}$, where $w^2 = v$, $x \in \mathbb{R}$ and $|x| < 1$. Since $h_{w,-iwx} = h^{-1}_{\bar{w},iwx}$, the map $h_{\bar{w},iwx}$ restricted to S^1 is an orientation preserving conjugacy of g and $h_{u,v}$, so g and $h_{u,v}$ have the same rotation number. After some algebra, we get

$$\begin{aligned} g &\sim \begin{pmatrix} \bar{w} & iwx \\ -i\bar{w}x & w \end{pmatrix} \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \begin{pmatrix} w & -iwx \\ i\bar{w}x & \bar{w} \end{pmatrix} \\ &= \begin{pmatrix} u|v| + 2i|v|^2x - \bar{u}|v|x^2 & |v|^2x^2 + 2\,\mathrm{Im}(u)|v|x + |v|^2 \\ |v|^2x^2 + 2\,\mathrm{Im}(u)|v|x + |v|^2 & \bar{u}|v| - 2i|v|^2x - u|v|x^2 \end{pmatrix} \\ &\sim \begin{pmatrix} u + 2i|v|x - \bar{u}x^2 & |v|x^2 + 2\,\mathrm{Im}(u)x + |v| \\ |v|x^2 + 2\,\mathrm{Im}(u)x + |v| & \bar{u} - 2i|v|x - ux^2 \end{pmatrix}, \end{aligned}$$

the last expression being valid since $|v| \neq 0$. For convenience of notation let us write $c := |v|$, so that

$$g \sim \begin{pmatrix} u + 2icx - \bar{u}x^2 & cx^2 + 2\,\mathrm{Im}(u)x + c \\ cx^2 + 2\,\mathrm{Im}(u)x + c & \bar{u} - 2icx - ux^2 \end{pmatrix}.$$

Now we want to choose x so that g is a rotation. This means that the off-diagonal elements $cx^2 + 2\,\mathrm{Im}(u)x + c$ must be zero, and the corresponding solution for x is

$$x = \frac{-\mathrm{Im}(u) \pm \sqrt{(\mathrm{Im}(u))^2 - c^2}}{c}. \tag{1}$$

Since $|\mathrm{Im}(u)| > |v| = c$, this is a real number. By the triangle inequality, we have $|\mathrm{Im}(u)| - \sqrt{(\mathrm{Im}(u))^2 - c^2} < c$. Since we want $|x| < 1$, this means we should choose $+$ in (1) if $\mathrm{Im}(u) > 0$ and $-$ if $\mathrm{Im}(u) < 0$. For this choice of x, g is just multiplication by the complex number $\lambda/\bar{\lambda}$, where

$$\lambda := u + 2icx - \bar{u}x^2 = (1 - x^2)\,\mathrm{Re}(u) + i((1 + x^2)\,\mathrm{Im}(u) + 2cx). \tag{2}$$

Thus to find the rotation number, we must find $\arg(\lambda/\bar{\lambda}) = 2\arg(\lambda)$. Since x satisfies $x^2 + 2(\mathrm{Im}(u)/c)x + 1 = 0$, we have

$$1 + x^2 = -2\frac{\mathrm{Im}(u)}{c}x. \tag{3}$$

We also have $1 - x^2 = -2x^2 - 2(\mathrm{Im}(u)/c)x = -2x(x + \mathrm{Im}(u)/c)$. Moreover, $x + \mathrm{Im}(u)/c = \pm\sqrt{(\mathrm{Im}(u))^2 - c^2}/c$, where we choose $+$ if $\mathrm{Im}(u) > 0$ and $-$ if $\mathrm{Im}(u) < 0$. Thus

$$1 - x^2 = -2x\,\mathrm{sgn}(\mathrm{Im}(u))\frac{\sqrt{(\mathrm{Im}(u))^2 - c^2}}{c}. \tag{4}$$

Substituting (3) and (4) into (2) and simplifying gives

$$\lambda = -\frac{2x}{c}\sqrt{(\mathrm{Im}(u))^2 - c^2}\,(\mathrm{sgn}(\mathrm{Im}(u))\,\mathrm{Re}(u) + i\sqrt{(\mathrm{Im}(u))^2 - c^2}).$$

It follows that

$$\tan(\tfrac{1}{2}\arg(\lambda/\bar{\lambda})) = \tan(\arg(\lambda)) = \mathrm{sgn}(\mathrm{Im}(u))\sqrt{(\mathrm{Im}(u))^2 - c^2}/\mathrm{Re}(u).$$

Since we are assuming $(\mathrm{Im}(u))^2 - c^2 = (\mathrm{Im}(u))^2 - |v|^2 \geq 0$, we can replace $\sqrt{(\mathrm{Im}(u))^2 - c^2}$ in the above expression with $\mathrm{Re}(\sqrt{(\mathrm{Im}(u))^2 - |v|^2})$. Thus if g is the rotation by ρ (that is, multiplication by $e^{i2\pi\rho}$) then ρ is also (up to an integer) the rotation number of $h_{u,v}|_{S^1}$ and satisfies

$$\tan(\pi\rho) = \frac{\mathrm{sgn}(\mathrm{Im}(u))}{\mathrm{Re}(u)} \mathrm{Re}(\sqrt{(\mathrm{Im}(u))^2 - |v|^2}). \tag{5}$$

The reason for writing (5) in this way is that it remains valid when $(\mathrm{Im}(u))^2 - |v|^2 < 0$, since then ρ is an integer. Finally, in the case when $v = 0$, the map $h_{u,v}|_{S^1}$ is just multiplication by $u/\bar{u}$, so (5) is still correct. We summarize the preceding computations in a theorem.

THEOREM 6. *Let T be an orientation preserving Möbius transformation of the unit circle onto itself, and let $T(z)$ be prescribed in the form $T(z) = (uz + v)/(\bar{v}z + \bar{u})$ for constants u, v satisfying $|v| < |u|$. Then the rotation number ρ of T satisfies* (5).

Thus, if we want to compute $\rho(A)$ when $C = S^1$, we can do it in the following way. Write $M_j = \begin{pmatrix} i & -ia_j \\ i\bar{a}_j & -i \end{pmatrix}$ and compute $\begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} = M_n M_{n-1} \ldots M_1$. Then we can use (5) to determine the fractional part of $\rho(A)$. We have to use other methods to determine the integer part of $\rho(A)$, but in concrete situations this should not be too difficult.

We conclude this section by showing that every orientation preserving Möbius transformation of the unit circle is a projectivity. We start with a technical, but interesting lemma.

LEMMA 7. *Let $d \neq 1$ be a complex number of modulus 1, let*

$$a = \frac{\bar{d} + 1}{2}, \qquad b = 2\frac{\mathrm{Re}(d) + 1}{\mathrm{Re}(d) + 3}, \qquad c = \frac{d + 1}{2},$$

and $A = (a, b, c)$. Then the moduli of a, b, c are less than 1 and $f_A : S^1 \to S^1$ is the rotation by the angle $\pi + 2\arg((d - 1)(d + 3))$.

Proof. We have

$$|c|^2 = \frac{(d + 1)(\bar{d} + 1)}{4} = \frac{1 + d + \bar{d} + 1}{4} = \frac{\mathrm{Re}(d) + 1}{2}.$$

Therefore $|c| < 1$, and since $a = \bar{c}$, we also have $|a| < 1$. Moreover,

$$b = 2\frac{(\mathrm{Re}(d) + 1)/2}{(\mathrm{Re}(d) + 3)/2} = \frac{2|c|^2}{1 + |c|^2},$$

and thus $|b| < 1$.

Since $a = \bar{c}$ and b is real, we get

$$f_A \sim \begin{pmatrix} i & -ic \\ i\bar{c} & -i \end{pmatrix} \begin{pmatrix} i & -ib \\ ib & -i \end{pmatrix} \begin{pmatrix} i & -i\bar{c} \\ ic & -i \end{pmatrix} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix},$$

where $u = i(-1 + 2bc - c^2)$ and $v = i(c + \bar{c} - b - b|c|^2)$. We have

$$c + \bar{c} = \frac{d + 1 + \bar{d} + 1}{2} = \mathrm{Re}(d) + 1,$$

so $2|c|^2 = c + \bar{c}$. Thus,

$$v = i\left(2|c|^2 - \frac{2|c|^2}{1 + |c|^2}(1 + |c|^2)\right) = 0.$$

This means that f_A is just multiplication by $u/\bar{u}$, that is, rotation by the angle $2\arg(u)$.

To compute $\arg(u)$, note that

$$bc = \frac{2|c|^2 c}{1 + |c|^2} = \frac{(c + \bar{c})c}{1 + |c|^2} = \frac{c^2 + |c|^2}{1 + |c|^2}.$$

Therefore

$$\begin{aligned} u = i(-1 + 2bc - c^2) &= i\left(-1 + 2\frac{c^2 + |c|^2}{1 + |c|^2} - c^2\right) \\ &= i\frac{-1 + |c|^2 + c^2 - c^2|c|^2}{1 + |c|^2} = i\frac{1 - |c|^2}{1 + |c|^2}(c - 1)(c + 1) \\ &= it(d - 1)(d + 3), \end{aligned}$$

where

$$t = \frac{1 - |c|^2}{4(1 + |c|^2)}.$$

Thus we have $\arg(u) = \pi/2 + \arg((d-1)(d+3))$, so $2\arg(u) = \pi + 2\arg((d-1)(d+3))$. ∎

Note that the limit of $\pi + 2\arg((d-1)(d+3))$ as d tends to 1 with decreasing argument (resp. with increasing argument) is 0 (resp. 2π), and $\pi + 2\arg((d-1)(d+3))$ cannot be 0 or 2π. Therefore the set of all values of $\pi + 2\arg((d-1)(d+3))$ as $|d| = 1$ and $d \neq 1$ is equal to the interval $(0, 2\pi)$. Hence, we get the following corollary.

COROLLARY 8. *For every rotation T of S^1 other than the identity there exist points a, b, c with moduli less then 1 such that $f_{(a,b,c)} = T$.*

Of course the identity can be represented as $f_{(a,a)}$. As we noted already, every orientation preserving Möbius transformation of S^1 onto itself is a perspectivity followed by a rotation. This gives us the desired result.

COROLLARY 9. *Every orientation preserving Möbius transformation of S^1 onto itself is a projectivity via a sequence of four or fewer points.*

We suspect that in the above corollary "four" can be replaced by "three," but checking that would require tedious calculations.

4. ELLIPSES VERSUS OTHER BOUNDARIES

In the introduction we referred to a projectivity with a fixed point which apparently exhibited the phenomenon called "phase locking." This phenomenon exists in the general case when n is arbitrary and D has an arbitrary convex shape. Before establishing this, let us make some simple observations. Suppose $A := (a_1, a_2, \ldots, a_n) \in \rho^{-1}(k)$ for some integer k, and let $x_0 \in C$ be a fixed point of f_A so that $F_A(X_0) - k = X_0$ when $X_0 \in e^{-1}(x_0)$. If the graph of $F_A(X) - k$ crosses the diagonal $\{(X, X) | X \in \mathbb{R}\}$ at (X_0, X_0), then A is an interior point of $\rho^{-1}(k)$, because small changes in A will cause small changes in the graph of $F_A(X) - k$. Hence the graph will still cross the diagonal (f_A will still have a fixed point) and the rotation number, which must change continously with A, will remain equal to k. If the graph of $F_A(X) - k$ does not cross the diagonal, but nevertheless satisfies $F_A(X) - k \neq \mathrm{id}$, we can still find an interior point $\hat{A}$ of $\rho^{-1}(k)$. For example, if the graph of $F_A(X) - k$ lies above the diagonal, we can move a_1 slightly to a new location $\hat{a}_1$ so that $f_{\hat{a}_1}(x_0)$ lies slightly clockwise from $f_{a_1}(x_0)$, and hence $F_{\hat{a}_1}(X_0) < F_{a_1}(X_0)$. If we then define $\hat{A} := (\hat{a}_1, a_2, \ldots, a_n)$, we get $F_{\hat{A}}(X_0) - k < X_0$. Provide $\hat{a}_1$ is close enough to a_1, the graph of $F_{\hat{A}}(X) - k$ will cross the diagonal, and thus $\hat{A}$ will be an interior point of $\rho^{-1}(k)$.

THEOREM 10. *Let D be a nonempty open convex region in $\mathbb{R}^2$ bounded by a closed curve C. Then the function $\rho: D^n \to \mathbb{R}$ satisfies* $\text{int}(\rho^{-1}(k)) \neq \varnothing$ *for each integer $k \in \rho(D^n)$.*

Proof. Let $A \in \rho^{-1}(k)$, let $x_0 \in C$ be a fixed point of f_A, and let $x_1 := f_{a_1}(x_0)$. Then a new sequence $A' := (a_1', a_2, ..., a_n) \in \rho^{-1}(k)$ can be obtained by choosing a point a_1' different from a_1 in the interior of the chord x_0x_1 (note that x_0 is then a fixed point of $f_{A'}$). Now consider a point $y_0 \in C$ different from x_0 and x_1. In this case $f_{a_1}(y_0) \neq f_{a_1'}(y_0)$, and since the perspectivities $f_{a_2}, ..., f_{a_n}$ are one-to-one, we also have $f_A(y_0) \neq f_{A'}(y_0)$. Thus either $f_A \neq \text{id}$ or $f_{A'} \neq \text{id}$, and by the remarks preceding the theorem, we can find an interior point $\hat{A}$ of $\rho^{-1}(k)$. ∎

In the case when D is a convex region of arbitrary shape, it is helpful to use derivatives to analyze the behavior of f_A. We are specifically interested in computing the one-sided derivatives of the lifting F_A of f_A. In general, these derivatives will depend on the homeomorphism $h: S^1 \to C$, but for our purposes it will be most convenient to assume that h has been chosen so that for some positive constant, the distance $d(x, y)$ for each $x, y \in C$ is proportional by that constant to the actual length of the anticlockwise arc $\widehat{xy}$. Thus in particular, the ratio $d(x, y)/d(w, z)$ will be exactly equal to the ratio of the true lengths of the anticlockwise arcs $\widehat{xy}$ and $\widehat{wz}$. Also helpful will be the fact that, by the convexity of D, the boundary C has one-sided derivatives everywhere, and is in fact differentiable, except possibly at a countable number of points. This follows from analogous basic results on real convex functions.

Since $F_A = F_{a_n} \circ F_{a_{n-1}} \circ \cdots \circ F_{a_1}$, and since the F_{a_i} are strictly increasing, we can show that the one-sided derivatives of F_A exist by showing that the one-sided derivatives of the F_{a_i} exist. Let $x_0 \in C$. Since C has a one-sided derivatives at x_0, there exists a "forward tangent" to C at x_0; that is, a line tangent at x_0 to an arc of C which starts at x_0 and continues a short distance anticlockwise. In the following lemma, we will write $x_1 := f_a(x_0)$ and denote by α_0 the angle which this line makes with the chord x_0x_1, and denote by β_1 the angle which the corresponding forward tangent at x_1 makes with x_0x_1 (see Fig. 4). We shall write $|x - y|$ for the Euclidean distance between x and y.

LEMMA 11. *Let D be a nonempty open convex region in $\mathbb{R}^2$ bounded by a closed curve C, let $a \in D$, and let F_a be the lifting of the projectivity f_a. Suppose also that the homeomorphism $h: S^1 \to C$ is such that the distances $d(x, y)$ for $x, y \in C$ are all proportional by the same positive constant to the actual corresponding arc lengths on C. Then the one-sided derivatives of F_a exist at every point of $\mathbb{R}$. In particular, if $X_0 \in \mathbb{R}$ and $x_0 := e(X_0)$, then the derivative $F_a'^{+}(X_0)$ is given by*

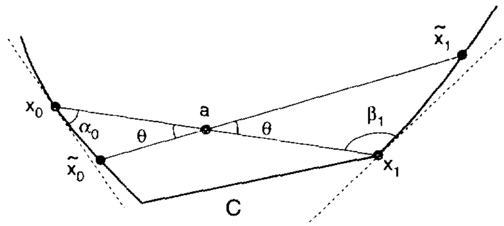

FIG. 4. Forward tangents to C.

$$F_a'^{+}(X_0) = \frac{|x_1 - a| \sin \alpha_0}{|x_0 - a| \sin \beta_1}, \tag{6}$$

where x_2, α_0, and β_1 are as defined above.

Proof. Let $X_0 \in \mathbb{R}$ and let $x_0 := e(X_0)$. We will show that the "forward derivative" $F_a'^{+}(X_0) := \lim_{X \downarrow X_0}(F_a(X) - F_a(X_0))/(X - X_0)$ is given by (6).

Let us use the notation in Fig. 4, and the symbols $\angle \tilde{x}_0 x_0 x_1$ and $\angle x_0 x_1 \tilde{x}_1$ to denote the corresponding angles at x_0 and x_1. We treat $\tilde{x}_0$, $\tilde{x}_1$, and θ as functions of $\tilde{X}_0$. Using the law of sines, and the proportionality between arc lengths on C and the distances $d(x, y)$, we get

$$\begin{aligned} F_a'^{+}(X_0) &= \lim_{\tilde{X}_0 \downarrow X_0} \frac{F_{a_i}(\tilde{X}_0) - F_a(X_0)}{\tilde{X}_0 - X_0} \\ &= \lim_{\tilde{X}_0 \downarrow X_0} \frac{|\tilde{x}_1 - x_1|}{|\tilde{x}_0 - x_0|} = \lim_{\tilde{X}_0 \downarrow X_0} \frac{|x_1 - a|}{|x_0 - a|} \frac{\sin(\angle \tilde{x}_0 x_0 x_1 + \theta)}{\sin(\angle x_0 x_1 \tilde{x}_1 + \theta)} \\ &= \frac{|x_1 - a|}{|x_0 - a|} \frac{\sin \alpha_0}{\sin \beta_1}, \end{aligned}$$

so (6) is proved. It should now be clear that a similar formula can be derived for the other one-sided derivative $F_a'^{-}(X_0)$. ∎

Despite the common feature of projectivities described in Theorem 10, there is an important difference in the behavior of projectivities defined on ellipses, as opposed to other boundaries of convex plane regions when A has three or more components a_i. We have seen in this case that if C is an ellipse and $\rho^{-1}(p/q)$ is nonempty for coprime natural numbers p and q, $q > 1$, then $f_A^q = \mathrm{id}$ for all $A \in D^n$ with $\rho(A) = p/q$. In fact, we conjecture that for any given n, p, and q as above, this feature characterizes ellipses. (Note that if $\rho(A) = p/q$, then $f_A^q = \mathrm{id}$ is equivalent to f_A being conjugate to a rotation. Indeed, if f_A is conjugate to a rotation then f_A^q is conjugate to the identity, so it is the identity. On the other hand, if f_A is not conjugate to a rotation then it has a point that is not periodic, and this point is not periodic for f_A^q.)

The following theorem proves the conjecture for the case $q = 2$.

THEOREM 12. *Let $n \geq 3$ and $k \geq 1$ be integers, let D be a nonempty open convex region in $\mathbb{R}^2$ bounded by a closed curve C, and suppose $\rho(A) = (2k + 1)/2$ for some $A \in D^n$. Then $f_A^2 = \mathrm{id}$ for all $A \in \rho^{-1}((2k + 1)/2)$ if and only if C is an ellipse.*

Before we start proving Theorem 12, we prove an auxiliary lemma. By a conic we mean a graph of an equation of degree 2.

LEMMA 13. *Assume that C is not a polygon and that it is locally a conic. Then C is an ellipse.*

Proof. That C is locally a conic means that for every $x \in C$ there is an open neighborhood C_x of x in C which is contained in a conic. We may assume that C_x is connected. By compactness, C can be covered by finite number of such curves. If one of the C_x's contains a segment of a straight line, it is either a segment or a union of two segments. Then any C_y intersecting C_x has to be of a similar form, and by finite induction we get that C is a polygon. Since we assumed it is not, this situation is impossible. Thus, every C_x is a piece of an ellipse, a parabola, or a hyperbola. Since those curves are analytic, any C_y intersecting C_x has to be a piece of the same conic. Again by finite induction, we see that the whole C is a subset of some conic. Since C is a closed curve, this conic has to be an ellipse and C has to be equal to it. ∎

Proof of Theorem 12. In view of Theorem 5, it will suffice to prove the "only if" part of Theorem 12, so assume that $f_A^2 = \mathrm{id}$ for all $A \in \rho^{-1}((2k + 1)/2)$. We want to use Lemma 13, so we start by showing that C cannot be a polygon. Suppose by way of contradiction that C is a polygon. Let $A := (a_1, a_2, ..., a_n) \in \rho^{-1}((2k + 1)/2)$. We may assume that there are at least two distinct a_i. Indeed, if all the a_i are identical then $\rho(A) = n/2$, and we can separate two of the a_i along a straight line without changing the rotation number and take this new configuration as our sequence A. Moreover, since the rotation number does not change if the indices of the a_i are cyclically permuted, we may also assume that $a_1 \neq a_n$, and locate a starting point x_0 at a vertex of C from which a_n appears to the left of a_1 as in Fig. 5a.

Since $f_A^2 = \mathrm{id}$, we can locate $2n$ corresponding points $x_1 := f_{a_1}(x_0)$, $x_2 := f_{a_2}(x_1)$, ..., $x_n := f_{a_n}(x_{n-1})$, $x_{n+1} := f_{a_1}(x_n)$, ... with $x_{2n} = x_0$. Note that the chord intersections $x_{i-1}x_i \cap x_{n+i-1}x_{n+i}$ for $i = 1, 2, ..., n$ must be nonempty

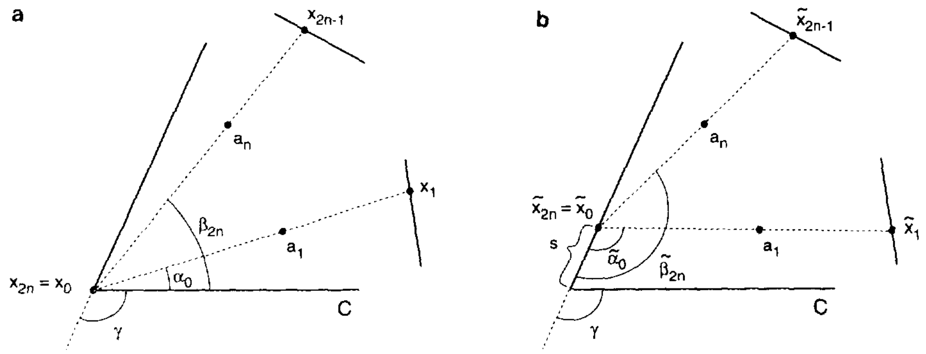

FIG. 5. (a), (b). Different starting points x_0 and $\tilde{x}_0$.

and nondisjoint from D, since the respective a_i belong to them. In fact, it will be convenient for us if each of these intersections is a single point in D, so that each pair $x_{i-1}x_i$, $x_{n+i-1}x_{n+i}$ is a pair of distinct chords. It will also be convenient if none of the points $x_1, x_2, ..., x_{2n-1}$ is a vertex of C.

If either of these conditions does not hold, then we can find a more suitable $A \in \rho^{-1}((2k + 1)/2)$ as follows. Assume first that $x_{n-1}x_n \neq x_{2n-1}x_{2n}$. In this case, to each of the points $x_i \in \{x_1, x_2, ..., x_{2n-1}\}$ assign an integer $m_i \in \{1, 2, ..., 2n - 1\}$ to represent the order in which the x_i occur, starting from x_0 and proceeding anticlockwise around C. In case $x_i = x_j$, we require that $m_i < m_j$ if $i < j$. Leaving x_0 and x_{2n} fixed, we obtain new versions of $x_1, x_2, ..., x_{2n-1}$ by choosing a small $\varepsilon > 0$ and moving each such x_i a distance of $m_i\varepsilon$ anticlockwise along C. We choose ε small enough so that no x_i moves to a different side of C and no new x_i is a vertex of C. It should be clear that the anticlockwise ordering of the new x_i, as represented by the numbers m_i, has not changed, and that $x_1, x_2, ..., x_{2n-1}$ are now pairwise distinct points. Moreover, the intersections $x_{i-1}x_i \cap x_{n+i-1}x_{n+i}$ are now single points in D for $i = 1, 2, ..., n$ (here we use our assumption that $x_{n-1}x_n \neq x_{2n-1}x_{2n}$). Hence we can locate new points $a_1, a_2, ..., a_n \in D$ at these respective intersections to obtain a new $A \in \rho^{-1}((2k + 1)/2)$.

On the other hand, if $x_{n-1}x_n = x_{2n-1}x_{2n}$, then either $x_{n-1} = x_{2n-1}$ and $x_n = x_{2n}$, or else $x_{n-1} = x_{2n}$ and $x_n = x_{2n-1}$. The first case is impossible, since $\rho(A)$ is not an integer. Therefore the second case occurs. Referring to Fig. 5a, we see that then $x_n \notin \{x_0, x_1\}$, so $x_0x_1 \neq x_nx_{n+1}$. Thus, in this case we assign the m_i to the points $x_1, x_2, ..., x_{2n-1}$ as before, except we break any ties $x_i = x_j$ by requiring that $m_i > m_j$ when $i < j$. We finish the adjustment by moving these points *clockwise* by a distance of $m_i\varepsilon$ and defining the new a_i as before. Since $x_0 = x_{2n}$, it should also be clear that the new $A \in D^n$ thus obtained also belongs to $\rho^{-1}((2k+1)/2)$. Indeed, we assume from now on that $x_1, x_2, \ldots, x_{2n-1}$ are pairwise distinct points, none of which is a vertex of C.

Now by assumption, $f_A^2 = \mathrm{id}$, so $(F_A^2)'^{+}(X) \equiv 1$ and, in particular, $(F_A^2)'^{+}(X_0) = 1$, where $X_0 \in e^{-1}(x_0)$. To see what this implies, let us use the notation α_{i-1} for each $i = 1, 2, \ldots, 2n$ to denote the angle that the forward tangent at x_{i-1} makes with the chord $x_{i-1}x_i$, and use β_i to denote the angle that the forward tangent at x_i makes with the chord $x_{i-1}x_i$. Let us also use the notation $a_{(i)}$ to represent a_i if $i \le n$ and a_{i-n} if $i > n$. Then applying the chain rule and Lemma 10 to the composite perspectivities F_{a_i}, we get

$$1 = (F_A^2)'^{+}(X_0) = \prod_{i=1}^{2n} \frac{|x_i - a_{(i)}|}{|x_{i-1} - a_{(i)}|} \frac{\sin \alpha_{i-1}}{\sin \beta_i}, \tag{7}$$

and this must hold for *any* starting point x_0.

This does not turn out to be the case, however. Using the same a_i, let us consider a different starting point $\tilde{x}_0 \in C$, located a short distance s clockwise from x_0 (see Fig. 5b). By assumption we must have $\tilde{x}_{2n} = \tilde{x}_0$, where $\tilde{x}_{2n}$ has the obvious meaning, and, since $x_1, x_2, \ldots, x_{2n-1}$ do not lie on vertices, we can choose s sufficiently small so that $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{2n-1}$ lie in the interiors of the same sides of C as their non-tilded counterparts. This means that the forward tangents at these points do not change direction, and hence, if s is small, the corresponding angles $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are very close to the previous values α_i and β_i for $i = 1, 2, \ldots, 2n-1$. Similarly, if s is small, the lengths $|\tilde{x}_i - a_{(i)}|$ and $|\tilde{x}_{i-1} - a_{(i)}|$ are very close to the previous values $|x_i - a_{(i)}|$ and $|x_{i-1} - a_{(i)}|$ for $i = 1, 2, \ldots, 2n$. Therefore, if $\tilde{X}_0 \in e^{-1}(x_0)$, only the factors $\sin \tilde{\alpha}_0$ and $1/\sin \tilde{\beta}_{2n}$ in the factorization of $(F_A^2)'^{+}(\tilde{X}_0)$ will be significantly different from their un-tilded counterparts in (7), because the instant we move away from the vertex, the direction of the forward tangent changes abruptly. Indeed, it should be clear that with the angle γ as indicated in Fig. 5b, we have

$$\begin{aligned}
\lim_{s \downarrow 0} \left(\frac{\sin \tilde{\alpha}_0}{\sin \tilde{\beta}_{2n}} - \frac{\sin \alpha_0}{\sin \beta_{2n}} \right) &= \frac{\sin(\alpha_0 + \gamma)}{\sin(\beta_{2n} + \gamma)} - \frac{\sin \alpha_0}{\sin \beta_{2n}} \\
&= \frac{\sin \beta_{2n}(\sin \alpha_0 \cos \gamma + \cos \alpha_0 \sin \gamma) - \sin \alpha_0(\sin \beta_{2n} \cos \gamma + \cos \beta_{2n} \sin \gamma)}{\sin \beta_{2n} \sin(\beta_{2n} + \gamma)} \\
&= \frac{\sin \gamma(\sin \beta_{2n} \cos \alpha_0 - \cos \beta_{2n} \sin \alpha_0)}{\sin \beta_{2n} \sin(\beta_{2n} + \gamma)} = \frac{\sin \gamma \sin(\beta_{2n} - \alpha_0)}{\sin \beta_{2n} \sin(\beta_{2n} + \gamma)},
\end{aligned}$$

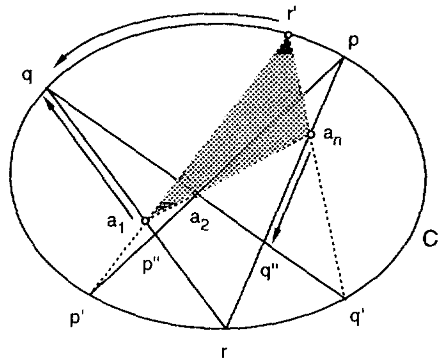

FIG. 6. The case $2k + 1 = n$.

which is clearly not zero, since $\beta_{2n} \neq \alpha_0$. It then follows that $(F_A^2)'^-(\tilde{X}_0) \not\to (F_A^2)'^+(X_0)$ as $s \downarrow 0$, contradicting the assumption that $(F_A^2)'^+(X) \equiv 1$. This proves that C cannot be a polygon.

To prove that C is an ellipse, let us first consider the case $2k + 1 = n$. For every point $x \in C$ there is a point $r \in C$ such that the chord xr is proper (i.e., disjoint from C except the endpoints). If we replace x by a nearby point $\tilde{x} \in C$ then the chord $\tilde{x}r$ is still proper. Therefore each $x \in C$ has a neighborhood $\widehat{pq} \subset C$ such that $r \notin \widehat{pq}$ and both chords pr and qr are proper (such an arc $\widehat{pq}$ appears in Fig. 6). By Lemma 13, in order to prove that C is an ellipse it is enough to show that in such a case the arc $\widehat{pq}$ is contained in a conic.

Choose points $p', q' \in C$ with p' between q and r, and q' between r and p so that p', q', and r are noncollinear as indicated, and draw the chords pp' and qq'. Let a_2 be the intersection of pp' and qq' (we write $a_2 := pp' \cap qq'$), and let $p'' := qr \cap pp'$ and $q'' := pr \cap qq'$. Also, if $k > 1$, let the $2k - 2$ points $a_3, \ldots, a_{n-1}$ satisfy $a_3 = \cdots = a_{n-1} = a_2$. For any arbitrary point $a_1 \in p''q$, let a_n be the point on pq'' collinear with a_1 and a_2, and let $A := (a_1, a_2, \ldots, a_n)$. Then a point $x_0 \in C$ collinear with the a_i will have period 2 with respect to f_A, and in terms of anticlockwise arcs, f_a^2 takes x_0 a total of $n = 2k + 1$ times around C. Hence $\rho(A) = (2k + 1)/2$, and by hypothesis, $f_A^2 = \mathrm{id}$, regardless of the original choice of $a_1 \in p''q$. This means that the point $r' \in C$, which belongs to the ray $p'a_1$, also belongs to the ray $q'a_n$, since $f_A^2(r') = r'$. Thus we can complete the inscribed hexagon in Fig. 6 with the chords $p'r'$ and $q'r'$ passing through a_1 and a_n, respectively.

In the diagram, solid lines represent fixed lines and dashed lines represent "movable" lines whose positions depend on the choice of a_1. The arrows in the diagram indicate that as a_1 slides from p'' to q along $p''q$, the point a_n slides from p to q'' along pq'', and r' slides from p to q along $\widehat{pq}$.

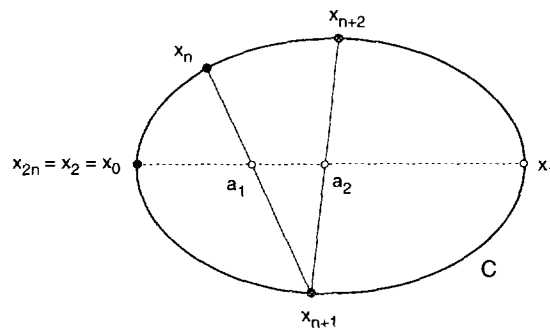

FIG. 7. The case $2k + 1 \neq n$.

This setup, including the "variable triangle" shaded in gray, satisfies the hypothesis of a theorem from projective geometry due to Braikenridge and MacClaurin (see [2]):

> *If the lines forming the sides of a variable triangle pass through three fixed noncollinear points p', q', a_2, while two vertices a_1 and a_n lie on fixed lines pr and qr not concurrent with $p'q'$, then the third vertex r' describes a conic.*

This is precisely what we wanted to show.

To finish the proof, suppose $2k + 1 \neq n$ and choose $A \in \rho^{-1}((2k + 1)/2)$. Since two a_i must then be distinct, we can cyclically reindex the a_i to obtain an $A \in \rho^{-1}((2k + 1)/2)$ with $a_1 \neq a_2$. We claim that this A can also be chosen so that if $x_0 \in C$ is a starting point collinear with a_1 and a_2 (closer to a_1), then $x_0 \neq x_{n+1}$ as in Fig. 7 (note that $x_1 \neq x_{n+1}$ since $\rho(A)$ is not an integer). To see this, suppose that $x_0 = x_{n+1}$. Then $x_0 = x_2 = x_{n+1} = x_{2n}$ and $x_1 = x_n = x_{n+2}$. In this case the chords x_0x_1, x_1x_2, x_nx_{n+1}, and $x_{n+1}x_{n+2}$ would all coincide, and we could slide a_1 along this chord, obtaining new sequences A for which the above points would remain fixed and $\rho(A) = (2k + 1)/2$. In particular, we could make $a_1 = a_2$, thereby reducing the number of distinct a_i while $\rho(A)$ remained the same. We could then reindex so that $a_1 \neq a_2$ and pick a new $x_0 \in C$ collinear with these two. If we again had $x_0 = x_{n+1}$, then the whole process could be repeated, reducing the number of distinct a_i even further. But this process must stop before all the a_i are collinear, for then we would have both $\rho(A) = (2k + 1)/2$ and $\rho(A) = n/2$. Hence our claim is proved, and we may assume that the five points x_0, x_1, x_n, x_{n+1}, and x_{n+2} are all distinct as in Fig. 7.

The ordering of those points can be the reverse of that depicted in Fig. 7. However, we can reduce that case to the previous one by an affine transformation with negative determinant (or by a mirror). Therefore we can assume that the ordering of the points x_i is exactly as in Fig. 7.

We will prove that in the above situation the arc $\widehat{x_{n+1}x_{n+2}}$ is contained

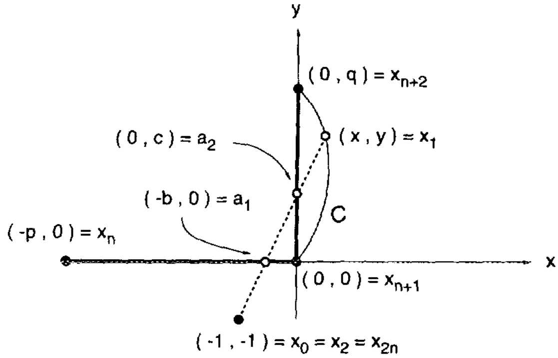

FIG. 8. A convenient projection of Fig. 7.

in a conic. However, in order to be able to use Lemma 13, we have to show that we can find such an arc containing an arbitrary (previously chosen) point $x \in C$ in its interior. If we replace the points $x_0, x_1, \ldots, x_{2n-1}$ by $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{2n-1}$ ordered along C in the same way, and then place the points $\tilde{a}_i$ at the corresponding chords, then we get $\tilde{A}$ with $\rho(\tilde{A}) = \rho(A)$. Each $\tilde{a}_i$ has to belong to two specific chords, but they intersect each other since the corresponding chords for the x_i's did. The only thing we have to be careful about is whether our new chords are proper. This is definitely the case if the new points $\tilde{x}_i$ lie at the vertices of a convex polygon inscribed into C. Since C is not a polygon, by Lemma 1 we can inscribe a convex $(2n + 1)$-gon into C. Then we can place $\tilde{x}_i$'s in the desired order at some of its vertices in such a way that our point x is in the interior of the arc $\widehat{x_{n+1}x_{n+2}}$. Thus Lemma 13 can be used, so it remains to prove that in the situation depicted in Fig. 7, $\widehat{x_{n+1}x_{n+2}}$ is contained in a conic.

Note that since x_{n+2}, x_0, and x_{n+1} lie on vertices of a polygon inscribed in C, the chord x_0x_1 will lie entirely in D (except for x_0 and x_1) as long as x_1 is located somewhere in the interior of $\widehat{x_{n+1}x_{n+2}}$ as in Fig. 7. Thus if we slide x_1 along this arc while keeping x_0 fixed and keeping a_1 and a_2 at the chord intersections indicated in the figure, x_0 will remain a period two point of f_A and $\rho(A)$ will not change (think of the dashed line and the unshaded points as being movable). We will show that x_1 traces a conic during this process, and it will be convenient to first affinely project the whole picture in Fig. 7 onto a coordinate system as indicated in Fig. 8 (observe that for convenience, $x_0 = (-1, -1)$ and x_nx_{n+1}, $x_{n+1}x_{n+2}$ lie on the coordinate axes). Since the projection and its inverse preserve conics, it will suffice to do the proof in this coordinate system.

We will use forward derivatives to do the proof. Observe that as x_1 moves along C, the only forward tangent angles α_i, β_i (at corresponding points

x_i) which will change are α_0, α_1, β_1, and β_2. The corresponding factors in the forward derivative formula (7) which will change are sin α_0, sin α_1, 1/sin β_1, and 1/sin β_2. However, since we require a_1 and a_2 to remain collinear with $x_0(= x_2)$ and x_1, we have $\sin \alpha_0 = \sin \beta_2$ and $\sin \alpha_1 = \sin \beta_1$, so these four factors have a constant product of 1 and we can ignore them. There are exactly four other factors in (7) which will change, and since we are assuming that $f_A^2 = \mathrm{id}$ for each corresponding A, the product of these factors must be constant. Specifically, we have

$$\frac{|x_1 - a_1|}{|x_0 - a_1|}\frac{|x_2 - a_2|}{|x_1 - a_2|}\frac{|x_{n+1} - a_1|}{|x_n - a_1|}\frac{|x_{n+2} - a_2|}{|x_{n-1} - a_2|} = K, \tag{8}$$

where K is a positive constant.

By similar triangles, the positive numbers b and c indicated in Fig. 8 satisfy both $c/b = y/(x + b)$ and $c/b = (1 + c)/1$. Solving for b and c gives $b = (y - x)/(y + 1)$ and $c = (y - x)/(x + 1)$, so that $a_1 = ((x - y)/(y + 1), 0)$ and $a_2 = (0, (y - x)/(x + 1))$. Then with the positive constants p and q as labeled in the figure, it is straightforward to compute that (8) implies

$$\frac{y^2}{x^2}\frac{(q + qx + x - y)^2}{(p + py + x - y)^2} = K^2. \tag{9}$$

Before taking the square root of both sides of (9), let us observe that x, $y > 0$ on the anticlockwise arc $\widehat{x_{n+1}x_{n+2}}$. The expressions in parentheses are also positive. For instance, by comparing the slopes of x_0x_{n+2} and x_0x_1 in Fig. 8, we get $(q + 1)/1 > (y + 1)/(x + 1)$. Therefore $q + qx + x - y > 0$, and since $p + py - y + x$ is continuous with respect to x and y, it must therefore be of constant sign to satisfy (9). By letting $x, y \to 0$, we see that this sign is positive, so we can take the square root of both sides of (9) and rearrange to get the second-degree equation $y(q + qx + x - y) = Kx(p + py + x - y)$. It follows that $\widehat{x_{n+1}x_{n+2}}$ is contained in a conic, and we are done. ∎

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